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THE AXISYMMETRIC END PROBLEM FOR TRANSVERSELY ISOTROPIC CIRCULAR CYLINDERS†

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Abstract—Energy-decay inequalities are applied in investigating the decay of end effects in a transversely isotropic circular cylinder subject to torsionless axisymmetric end loads. A lower bound (in terms of the elastic constants) is obtained for the rate of exponential decay of stresses and this is compared with results of other authors. For a highly anisotropic medium, a slow decay rate is predicted thus anticipating disagreement with Saint-Venant's principle in this case.

1. INTRODUCTION

This work is concerned with investigating the issue of Saint-Venant's principle for the torsionless axisymmetric problem for an anisotropic elastic circular cylinder. The main purpose is to assess the influence of anisotropy on the decay of "end effects." This matter would seem to be of particular interest in view of the widespread current activity in research on elasticity problems involving highly anisotropic and composite media.

Saint-Venant's principle in isotropic elasticity theory has received considerable attention in recent years (see, e.g. [1–6] and the references cited there.) These works have been concerned with establishing the exponential decay of stresses away from that portion of the boundary of an elastic solid which is subject to self-equilibrated surface tractions. The main purpose in these investigations has been to establish an exponential decay inequality for the stress components and to determine the decay constant explicitly. This decay constant then provides a lower bound for the actual rate of decay.

For the plane problem of elasticity, the effect of anisotropy was examined recently in [7]. In particular, it was shown that the exponential decay rate (which depends on the elastic constants) is always less than that obtained in [2] for the isotropic case. Furthermore, the results suggested that end effects may undergo *slow* decay for highly anisotropic media. In a subsequent work[8] the analysis was applied to the plane problem for a transversely isotropic medium with a high degree of anisotropy. Such transversely isotropic materials have had wide usage as models for fiber-reinforced composites (see, e.g. [9–11]). In the limit of small extensibility and compressibility,‡ disagreement with Saint-Venant's principle was anticipated, confirming similar observations made in [10–12].

In the present work, we consider analogous questions for the more complicated three-dimensional problem for a transversely isotropic circular cylinder under torsionless axisymmetric end load. The axis of elastic symmetry is along the cylinder axis. We note the relevance of this model to theories for fiber-reinforced composites[9]. For the *isotropic* cylinder,

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‡ This latter assumption is inessential to the analysis.

an exponential decay inequality for the stresses was established in [6], and the results compared with analyses of other authors. Similar arguments are used in the present study. The "exact solution" to the end problem for isotropic circular cylinders has been the subject of numerous investigations†—however, the corresponding problem for the transversely isotropic case appears to have been considered only in [16,17].

In the next section we state the boundary value problem to be considered in terms of stresses and displacements referred to cylindrical coordinates. Sections 3 and 4 are concerned with a reformulation of the problem in terms of a pair of stress functions. This enables us to adopt the arguments of [2,6] to the present case. The strain energy and a related quadratic functional are introduced in Sections 5 and 6. An energy decay inequality is established in Sections 7 and 8, which in turn leads to pointwise estimates for the stresses. Our main interest in this paper is in investigating the decay constant. In Section 9 we compare our results with those obtained in [17] for the two transversely isotropic materials magnesium and zinc.

Finally, in Section 10, we consider the case of highly anisotropic materials. In the limit as the degree of anisotropy increases, we show that the decay rate furnished by the analysis here (which is a lower bound on the actual rate of decay) tends to zero. Thus we anticipate disagreement with Saint-Venant's principle in the limit considered. We note that the asymptotic form of the decay rate is precisely that which would result from a limiting analysis of the exact rate given in [17].

2. BASIC EQUATIONS

We consider an elastic circular cylinder of radius c and length l and we use cylindrical coordinates r, θ, z. The cylinder is assumed to be transversely isotropic, with axis of elastic symmetry along the z-axis. Thus, if we consider the transversely isotropic cylinder to be a model for a fiber-reinforced composite[9] the fibers are in parallel alignment along the z-direction. For torsionless axisymmetric deformations of such a cylinder, the non-zero displacements, strains and stresses are respectively denoted by

$$u_r(r, z),\ u_z(r, z);\ e_{rr},\ e_{\theta\theta},\ e_{zz},\ e_{rz} \quad \text{and} \quad \tau_{rr},\ \tau_{\theta\theta},\ \tau_{zz},\ \tau_{rz}.$$

In the absence of body forces, the fundamental field equations of linear elasticity for torsionless axisymmetric deformations of the cylinder may be written as follows (see, e.g. [18,19]).

Equilibrium equations

$$r\frac{\partial \tau_{rr}}{\partial r} + r\frac{\partial \tau_{rz}}{\partial z} + \tau_{rr} - \tau_{\theta\theta} = 0, \tag{2.1}$$

$$r\frac{\partial \tau_{rz}}{\partial r} + r\frac{\partial \tau_{zz}}{\partial z} + \tau_{rz} = 0. \tag{2.2}$$

Stress–strain relations

$$\tau_{rr} = ae_{rr} + (a - 2\bar{\mu})e_{\theta\theta} + be_{zz}, \tag{2.3}$$

$$\tau_{\theta\theta} = ae_{\theta\theta} + (a - 2\bar{\mu})e_{rr} + be_{zz}, \tag{2.4}$$

$$\tau_{zz} = \bar{a}e_{zz} + b(e_{rr} + e_{\theta\theta}), \tag{2.5}$$

$$\tau_{rz} = \mu e_{rz} \tag{2.6}$$

† See, e.g. [13–15] for references to previous work.

Strain–displacement relations

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad re_{\theta\theta} = u_r, \tag{2.7}$$

$$e_{zz} = \frac{\partial u_z}{\partial z}, \qquad e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \tag{2.8}$$

In (2.3)–(2.6) we have used the notation for the five elastic constants a, $\bar{a}$, b, μ, $\bar{\mu}$ introduced by Eubanks and Sternberg in [18]. The special case of isotropy is found by letting

$$a = \bar{a} = \frac{2\mu(1-\sigma)}{1-2\sigma}, \qquad b = \frac{2\mu\sigma}{1-2\sigma}, \qquad \bar{\mu} = \mu, \tag{2.9}$$

where μ, σ denote the shear modulus and Poisson's ratio for an isotropic medium. The elastic constants introduced above are related to the usual engineering constants in the following manner:

$$\begin{aligned} a &= \frac{E_T(1 - \nu_{LT}^2 E_T/E_L)}{(1+\nu_{TT})(1-\nu_{TT}-2\nu_{LT}^2 E_T/E_L)}, \qquad \bar{a} = \frac{E_L(1-\nu_{TT})}{1-\nu_{TT}-2\nu_{LT}^2 E_T/E_L}, \\ b &= \frac{E_T \nu_{LT}}{1-\nu_{TT}-2\nu_{LT}^2 E_T/E_L}, \qquad \bar{\mu} = \frac{E_T}{2(1+\nu_{TT})} = G_T, \qquad \mu = G_{LT}, \end{aligned} \tag{2.10a}$$

where L denotes the direction parallel to the z-axis, T the transverse direction and ν, E, G denote Poisson's ratio, Young's Modulus and shear modulus respectively.

The relations inverse to (2.10a) are

$$\begin{aligned} E_T &= \frac{4\bar{\mu}(a\bar{a} - b^2 - \bar{a}\bar{\mu})}{a\bar{a} - b^2}, \qquad E_L = \frac{a\bar{a} - b^2 - \bar{a}\bar{\mu}}{a - \bar{\mu}}, \\ \nu_{TT} &= \frac{a\bar{a} - b^2 - 2\bar{a}\bar{\mu}}{a\bar{a} - b^2}, \qquad \nu_{LT} = \frac{b}{2(a-\bar{\mu})}. \end{aligned} \tag{2.10b}$$

As shown in [18], necessary and sufficient conditions for positive definiteness of the strain–energy density are that

$$a > 0, \quad \bar{a} > 0, \quad \mu > 0, \quad \bar{\mu} > 0, \quad a\bar{a} - b^2 - \bar{a}\bar{\mu} > 0, \tag{2.11}$$

or equivalently, on using (2.10),

$$E_T > 0, \quad E_L > 0, \quad G_{LT} > 0, \quad -1 < \nu_{TT} < 1 - 2\nu_{LT}^2 E_T/E_L. \tag{2.12}$$

We assume henceforth that (2.11), (2.12) hold.

We now consider the boundary conditions for a cylinder with prescribed axisymmetric normal and shear tractions on the end $z = 0$, with the remainder of the boundary traction free. Thus we obtain

$$r = c\colon \tau_{rz}(c, z) = \tau_{rr}(c, z) = 0, \qquad 0 \leqslant z \leqslant l; \tag{2.13}$$

$$z = 0\colon \tau_{zz}(r, 0) = f(r), \qquad \tau_{zr}(r, 0) = g(r), \qquad 0 \leqslant r \leqslant c; \tag{2.14}$$

$$z = l\colon \tau_{zz}(r, l) = \tau_{zr}(r, l) = 0, \qquad 0 \leqslant r \leqslant c. \tag{2.15}$$

A necessary condition for the existence of an equilibrium state is the vanishing of the total axial force:

$$2\pi \int_0^c rf(r)\,\mathrm{d}r = 0. \tag{2.16}$$

The given functions f and g are assumed continuously differentiable on $[0, c]$.

We consider twice continuously differentiable displacement fields satisfying the equations (2.1)–(2.8) subject to the boundary conditions (2.13)–(2.15). Since we are dealing with the solid cylinder, we must take account of differentiability conditions *on the z-axis*, that is as $r \to 0$. As shown in [6], it follows from (2.2) that

$$\tau_{rz}(r, z) = O(r) \quad \text{as} \quad r \to 0, \qquad 0 \leqslant z \leqslant l, \tag{2.17}$$

uniformly in z. Also (2.7) implies that

$$u_r(r, z) = O(r) \quad \text{as} \quad r \to 0, \qquad 0 \leqslant z \leqslant l, \tag{2.18}$$

uniformly in z. Furthermore, (2.17) and the second of (2.14) imply that

$$g(r) = O(r) \quad \text{as} \quad r \to 0. \tag{2.19}$$

Finally, to ensure continuity of τ_{rz} we require that

$$g(c) = 0. \tag{2.20}$$

It is assumed that the given functions f and g satisfy (2.16), (2.19), (2.20). In conclusion here we observe that the classical Kirchhoff uniqueness theorem of linear elasticity guarantees that the solution of the problem posed above is unique to within an axisymmetric rigid body displacement.

3. STRESS FUNCTIONS

For the *isotropic* torsionless axisymmetric problem, the best known stress function representation is that due to Love[20], involving a single stress function satisfying the (axisymmetric) biharmonic equation. However, the resulting expressions for the stress components, involving third derivatives of the stress function, are not in a convenient form for the type of analysis used here. In particular, it is desirable to integrate the boundary conditions (2.13)–(2.15). For the isotropic case considered in [6], we introduced a stress function representation involving two stress functions, both satisfying second order partial differential equations. Furthermore, the traction boundary conditions, involving second derivatives, were easily integrated. We adopt a similar procedure here.

For the transversely isotropic torsionless axisymmetric problem, Eubanks and Sternberg [18] have shown that a complete solution of (2.1)–(2.8) is given by

$$u_r = -(\mu + b)\Phi_{rz} \tag{3.1}$$

$$u_z = a\nabla^2\Phi + (\mu - a)\Phi_{zz}, \tag{3.2}$$

where $\Phi(r, z)$ satisfies

$$\nabla_1^2\nabla_2^2\Phi = 0. \tag{3.3}$$

Here ∇_i^2 denotes the operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + c_i^2\frac{\partial^2}{\partial z^2}, \tag{3.4}$$

where c_1^2, c_2^2 are the roots of the quadratic equation

$$x^2 + [(b^2 + 2b\mu - a\bar{a})/a\mu]x + \bar{a}/a = 0. \tag{3.5}$$

The roots of (3.5) may be real or complex, depending on the values of the elastic constants. In (3.1), (3.2) the subscripts denote partial derivatives and we adopt this notation from now on. For the special case of isotropy, $c_1^2 = c_2^2 = 1$ and the above stress function due to Lekhnitskii is equivalent to Love's stress function[20].

To establish the alternative stress function representation in the present instance, we adapt the argument of Love[20] for the isotropic case. Thus following [20] p. 274–276 we deduce that u_r and u_z can be represented in the form

$$u_r = -\frac{1}{2\bar{\mu}}(\varphi_r + \Omega_r), \qquad u_z = \left(\frac{1}{2\bar{\mu}} - \frac{1}{\mu}\right)\varphi_z + \frac{1}{2\bar{\mu}}\Omega_z, \tag{3.6}$$

where Ω and φ satisfy the differential equations

$$A\left(\varphi_{rr} + \frac{1}{r}\varphi_r\right) + B\varphi_{zz} = \Omega_{zz}, \tag{3.7}$$

$$\Omega_{rr} + \frac{1}{r}\Omega_r + \frac{(1-\bar{\nu})}{A}\Omega_{zz} = \varphi_{zz}\left[\nu_{TT} - 1 + \frac{(1-\bar{\nu})B}{A}\right]. \tag{3.8}$$

Here we have introduced the notation

$$\bar{\nu} = \nu_{LT}E_T/E_L, \qquad A = 2\bar{\mu}\left(\frac{1}{E_L} - \frac{\bar{\nu}^2}{E_T}\right), \qquad B = \frac{2\bar{\mu}}{\mu} - 1 - \bar{\nu}, \tag{3.9}$$

where by virtue of (2.12), the constant A is positive.

For our purposes here, we define a new function χ such that

$$-\chi_z = r\Omega_r, \tag{3.10}$$

$$-\chi_r = -\frac{(1-\bar{\nu})}{A}r\Omega_z + \left[\nu_{TT} - 1 + \frac{(1-\bar{\nu})B}{A}\right]r\varphi_z. \tag{3.11}$$

Equation (3.8) ensures that $\chi_{zr} = \chi_{rz}$. Thus, we rewrite (3.6) in the form

$$ru_r = \frac{1}{2\bar{\mu}}(\chi_z - r\varphi_r), \qquad ru_z = \frac{1}{2\bar{\mu}}\left[\frac{A}{1-\bar{\nu}}\chi_r - \left\{\frac{A(1-\nu_{TT})}{1-\bar{\nu}} + \bar{\nu}\right\}r\varphi_z\right]. \tag{3.12}$$

The stresses following from (3.12) and (2.3)–(2.8) are given by

$$r^2\tau_{rr} = r^2\varphi_{zz} + r\varphi_r - \chi_z, \tag{3.13}$$

$$r^2\tau_{\theta\theta} = \nu_{TT}r^2\varphi_{zz} + \bar{\nu}r^2\varphi_{rr} - (1-\bar{\nu})r\varphi_r + \chi_z, \tag{3.14}$$

$$r\tau_{zz} = (r\varphi_r)_r, \tag{3.15}$$

$$\tau_{rz} = -\varphi_{rz}. \tag{3.16}$$

where use has been made of equations (3.17), (3.18) below.

We make special note of the simplicity of the foregoing relations. In particular the representations (3.13), (3.15), (3.16) are exactly the same as those obtained in [6] for the isotropic case. The close analogy between (3.15), (3.16) and the corresponding formulae in terms of

Airy's stress function in plane elasticity was also noted in [6]. From (3.7), (3.10), (3.11) we find that the governing equations for the functions φ, χ are

$$\varphi_{rr}+\frac{1}{r}\varphi_r+\left(\frac{1-\nu_{TT}}{1-\bar{\nu}}\right)\varphi_{zz}=\frac{1}{1-\bar{\nu}}\frac{\chi_{rz}}{r}, \tag{3.17}$$

$$r\chi_{rr}-\chi_r+\left(\frac{1-\bar{\nu}}{A}\right)r\chi_{zz}=-\left[\nu_{TT}-1+\frac{(1-\bar{\nu})B}{A}\right]r^2\varphi_{zr}. \tag{3.18}$$

The isotropic case is recovered by setting $\nu_{TT}=\bar{\nu}=\sigma$, $\bar{\mu}=\mu$ and we obtain the representation

$$ru_r=\frac{1}{2\mu}(\chi_z-r\varphi_r), \qquad ru_z=\frac{1}{2\mu}(\chi_r-r\varphi_z), \tag{3.19}$$

while (3.17), (3.18) become

$$(1-\sigma)r\nabla^2\varphi=\chi_{rz}, \tag{3.20}$$

$$r\chi_{rr}-\chi_r+r\chi_{zz}=0. \tag{3.21}$$

The stress function representation (3.19)–(3.21) was used extensively in [6]. The present stress functions do not appear to have been previously presented in the explicit form given here.

We remark that the completeness of the above stress functions may be established by using Love's argument; alternatively, φ, χ may be expressed in terms of Lekhnitskii's stress function Φ which is known to be complete[18].

Finally here we note that the most general axisymmetric rigid body displacement is given by the special forms

$$\varphi=\alpha+\beta\log r+\gamma z, \qquad \chi=\beta z+\delta+\eta r^2, \tag{3.22}$$

where α, β, γ, δ, η are constants. These expressions are exactly the same as those obtained in [6] for the isotropic case.

4. THE BOUNDARY VALUE PROBLEM FOR φ, χ

By virtue of the fact that the expressions (3.13), (3.15), (3.16) for τ_{rr}, τ_{zz}, τ_{rz} do not involve the elastic constants, the boundary conditions (2.13)–(2.15), when written in terms of φ, χ are formally equivalent to those obtained in [6] for the isotropic case. Thus the integration procedure used in [6] may be followed here also and we refer to that reference for details. On integration of the boundary conditions, the constants of integration thereby introduced are eliminated by adding an appropriate rigid body displacement of the form (3.22). In this way, the final form of the boundary value problem reads as follows. The functions φ, χ must satisfy (3.17), (3.18) subject to the boundary conditions

$$r=c\colon \varphi_r=0, \qquad 0\leqslant z\leqslant l, \tag{4.1}$$

$$r=c\colon c^2\varphi_z-\chi=0, \qquad 0\leqslant z\leqslant l, \tag{4.2}$$

$$z=l\colon \varphi=0, \qquad 0\leqslant r\leqslant c, \tag{4.3}$$

$$z=l\colon \varphi_z=0, \qquad 0\leqslant r\leqslant c, \tag{4.4}$$

$$z=0\colon \varphi_z=G(r), \qquad \varphi_r=F(r), \qquad 0\leqslant r\leqslant c. \tag{4.5}$$

$$r\to 0\colon \varphi_r=O(r), \qquad \varphi_z=O(1), \qquad \chi_r=O(r), \qquad \chi_z=O(r^2), \qquad \varphi=O(1), \tag{4.6}$$

uniformly in z for $0 \leqslant z \leqslant l$. The functions F and G are given by

$$F(r) = \frac{1}{r}\int_0^r \rho f(\rho)\,\mathrm{d}\rho, \tag{4.7}$$

$$G(r) = C - \int_0^r g(\rho)\,\mathrm{d}\rho, \tag{4.8}$$

where

$$C = \int_0^c \left(1 + \frac{1-\nu_{TT}}{1+\nu_{TT}}\frac{r^2}{c^2}\right) g(r)\,\mathrm{d}r. \tag{4.9}$$

We assume that a solution to the above boundary value problem exists. Then, on integrating (3.17) with respect to r from $r = 0$ to $r = c$ for fixed z and using (4.1), (4.6) we obtain

$$(1-\nu_{TT})\int_0^c r\varphi_{zz}(r, z)\,\mathrm{d}r = \chi_z(c, z).$$

On using (4.2), this may be written

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(1-\nu_{TT})\int_0^c r\varphi(r, z)\,\mathrm{d}r - c^2\varphi(c, z)\right] = 0. \tag{4.10}$$

Integrating with respect to z and using (4.3), (4.4), we find that

$$(1-\nu_{TT})\int_0^c r\varphi(r, z)\,\mathrm{d}r - c^2\varphi(c, z) = 0, \qquad 0 \leqslant z \leqslant l. \tag{4.11}$$

This "conservation property" of the solution is completely analogous to that obtained in [6].

5. STRAIN ENERGY REPRESENTATION

The strain energy $U(\zeta)$ contained in the portion of the cylinder for which $\zeta \leqslant z \leqslant l$ is given by

$$U(\zeta) = 2\pi \int_\zeta^l \int_0^c Wr\,\mathrm{d}r\,\mathrm{d}z, \qquad 0 \leqslant \zeta \leqslant l, \tag{5.1}$$

where W is given by [18]

$$2W = \frac{1}{E_T}(\tau_{rr}^2 + \tau_{\theta\theta}^2) + \frac{1}{E_L}\tau_{zz}^2 + \frac{1}{\mu}\tau_{rz}^2 - \frac{2\bar{\nu}}{E_T}\tau_{zz}(\tau_{rr} + \tau_{\theta\theta}) - \frac{2\nu_{TT}}{E_T}\tau_{rr}\tau_{\theta\theta}. \tag{5.2}$$

Alternatively, we have the representation

$$2U(\zeta) = -2\pi \int_0^c [\tau_{zr}(r, \zeta)u_r(r, \zeta) + \tau_{zz}(r, \zeta)u_z(r, \zeta)]r\,\mathrm{d}r, \tag{5.3}$$

which follows from the usual work-energy relation. We now express $U(\zeta)$ in terms of the stress functions φ, χ.

On substituting from (3.12), (3.15), (3.16) we obtain from (5.3)

$$-\frac{2\bar{\mu}}{\pi}U(z) = \int_0^c \left[\varphi_{rz}(r\varphi_r - \chi_z) + (r\varphi_r)_r\left\{\frac{A}{1-\bar{\nu}}\frac{\chi_r}{r} - \varphi_z\left(\frac{A(1-\nu_{TT})}{1-\bar{\nu}} + \bar{\nu}\right)\right\}\right]\mathrm{d}r. \tag{5.4}$$

From (5.4) it follows that

$$-\frac{2\bar{\mu}}{\pi}U'(z)=\int_0^c\left[\varphi_{rzz}(r\varphi_r-\chi_z)+\varphi_{rz}(r\varphi_{rz}-\chi_{zz})+(r\varphi_{rz})_r\left\{\frac{A}{1-\bar{\nu}}\frac{\chi_r}{r}-\varphi_z\left(\frac{A(1-\nu_{TT})}{1-\bar{\nu}}+\bar{\nu}\right)\right\}\right.$$
$$\left.+(r\varphi_r)_r\left\{\frac{A}{1-\bar{\nu}}\frac{\chi_{rz}}{r}-\varphi_{zz}\left(\frac{A(1-\nu_{TT})}{1-\bar{\nu}}+\bar{\nu}\right)\right\}\right]dr, \tag{5.5}$$

where the prime denotes differentiation with respect to z. On integration by parts and using the boundary conditions (4.1), (4.2), (4.6) and the differential equations (3.17), (3.18) we find that

$$-\frac{2\bar{\mu}}{\pi}U'(z)=-c^2\varphi_{zz}^2(c,z)$$
$$+\int_0^c\left[(1-\nu_{TT})r\varphi_{zz}^2+\frac{A}{r}(r\varphi_r)_r^2+\frac{2\bar{\mu}}{\mu}r\varphi_{rz}^2-2\bar{\nu}(r\varphi_r)_r\varphi_{zz}\right]dr. \tag{5.6}$$

Since $U(l)=0$, integration of (5.6) gives

$$\frac{2\bar{\mu}}{\pi}U(\zeta)=\int_\zeta^l h(z)\,dz, \tag{5.7}$$

where $h(z)$ denotes the expression on the right hand side of (5.6). We note that the stress function χ does not appear in (5.6).

6. AN AUXILIARY QUADRATIC FUNCTIONAL

To establish the exponential decay inequality for $U(\zeta)$, it is more convenient to deal with a quadratic functional closely related to the strain energy $U(\zeta)$ discussed above. We define $V(\zeta)$ by

$$\frac{2\bar{\mu}}{\pi}V(\zeta)=\int_\zeta^l\left\{-c^2\varphi_{zz}^2(c,z)+(1-\nu_{TT})\int_0^c\left[r\varphi_{zz}^2+\frac{A}{1-\nu_{TT}}\frac{1}{r}(r\varphi_r)_r^2+2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)r\varphi_{rz}^2\right]dr\right\}dz \tag{6.1}$$

for $0\leqslant\zeta\leqslant l$. From (5.6), (5.7) we see that when $\bar{\nu}=0$, $U(\zeta)\equiv V(\zeta)$; the relationship between U and V for non-zero $\bar{\nu}$ will be examined later in this section.

We now introduce a convenient scalar product notation for functions continuous on the interval $0\leqslant r\leqslant c$. For any two such functions f and g, we define

$$(f,g)=-c^2f(c)g(c)+(1-\nu_{TT})\int_0^c rf(r)g(r)\,dr. \tag{6.2}$$

By virtue of (4.11), the stress function φ satisfies the condition

$$(\varphi,1)=0,\qquad 0\leqslant z\leqslant l. \tag{6.3}$$

As pointed out in [6], the inequality $(f,f)\geqslant 0$ will *not* hold in general for arbitrary continuous functions f. However, the following alternative conditions do ensure the positivity of (f,f): (a) If $f(c)=0$, then $(f,f)\geqslant 0$ with equality if, and only if, $f\equiv 0$. (b) If $(f,1)=0$, then $(f,f)\geqslant 0$ with equality if, and only if, $f\equiv 0$. The result (a) is obvious from (6.2), while

(b) may be established by using the Schwarz inequality and the fact that $-1 < \nu_{TT} < 1$, as is implied by (2.12).

Expressed in terms of the scalar product (6.2), the quadratic functional V takes the form

$$\frac{2\bar{\mu}}{\pi} V(\zeta) = \int_\zeta^l \left[(\varphi_{zz}, \varphi_{zz}) + 2\left(\frac{\bar{\mu}/\mu - \bar{\nu}}{1-\nu_{TT}}\right)(\varphi_{rz}, \varphi_{rz}) + \int_0^c \frac{A}{r}(r\varphi_r)_r^2 \, dr\right] dz. \tag{6.4}$$

Since (6.3) implies that

$$(\varphi_z, 1) = (\varphi_{zz}, 1) = 0, \qquad 0 \leqslant z \leqslant l, \tag{6.5}$$

and since φ_{zr} vanishes at $r = c$, we see that a sufficient condition to ensure that V is positive† is that

$$\bar{\mu}/\mu - \bar{\nu} > 0. \tag{6.6}$$

We assume that the elastic constants are such that (6.6) holds.

The following three properties of V, which hold for $0 \leqslant z \leqslant l$, play a central role in the analysis to follow.

(i) $$\frac{2\bar{\mu}}{\pi} V'(z) = -(\varphi_{zz}, \varphi_{zz}) - 2\left(\frac{\bar{\mu}/\mu - \bar{\nu}}{1-\nu_{TT}}\right)(\varphi_{rz}, \varphi_{rz}) - A\int_0^c \frac{1}{r}(r\varphi_r)_r^2 \, dr, \tag{6.7}$$

(ii) $$\frac{2\bar{\mu}}{\pi}\int_z^l V(\zeta)\, d\zeta = (\varphi_z, \varphi_z) - (\varphi, \varphi_{zz}) + \left(\frac{\bar{\mu}/\mu - \bar{\nu}}{1-\nu_{TT}}\right)(\varphi_r, \varphi_r), \tag{6.8}$$

(iii) $$\left(1 - \frac{|\bar{\nu}|}{(1-\nu_{TT})\alpha}\right) V(z) \leqslant U(z) \leqslant \left(1 + \frac{|\bar{\nu}|}{(1-\nu_{TT})\alpha}\right) V(z). \tag{6.9}$$

In (6.9) the positive constant α is defined by

$$(1-\nu_{TT})\alpha = \min\left(\bar{\mu}/\mu - \bar{\nu}, [A(1-\nu_{TT})]^{1/2}\right). \tag{6.10}$$

Property (i) follows immediately from (6.4) on differentiation with respect to ζ. Property (ii) will be established from (6.7), on expressing the right hand side of that equation as a second derivative with respect to z. It is readily verified that (c.f. [6])

$$(\varphi_{zz}, \varphi_{zz}) = (\varphi_z, \varphi_z)'' - (\varphi, \varphi_{zz}) + (\varphi, \varphi_{zzzz}) \tag{6.11}$$

and

$$(\varphi_{rz}, \varphi_{rz}) = \tfrac{1}{2}(\varphi_r, \varphi_r)'' + (1-\nu_{TT})\int_0^c \varphi(r\varphi_{rzz})_r \, dr. \tag{6.12}$$

Furthermore, integration by parts and use of the differential equations and boundary conditions satisfied by φ, χ can be used to show that

$$(\varphi, \varphi_{zzzz}) = -2(\bar{\mu}/\mu - \bar{\nu})\int_0^c \varphi(r\varphi_{rzz})_r \, dr - A\int_0^c \frac{1}{r}(r\varphi_r)_r^2 \, dr. \tag{6.13}$$

Thus (6.7) may be written

$$\frac{2\bar{\mu}}{\pi} V'(z) = -\left[(\varphi_z, \varphi_z) - (\varphi, \varphi_{zz}) + \left(\frac{\bar{\mu}/\mu - \bar{\nu}}{1-\nu_{TT}}\right)(\varphi_r, \varphi_r)\right]''. \tag{6.14}$$

† We recall from the definition (3.9) that A is positive.

Integrating twice with respect to z and using the boundary conditions for φ at $z = l$ then yields the result (6.8).

Finally, we consider the result (6.9). From (5.6) and (6.7) we obtain

$$-\frac{2\bar{\mu}}{\pi} U'(z) + \frac{2\bar{\mu}}{\pi} V'(z) = \frac{2\bar{\nu}}{1-\nu_{TT}} (\varphi_{rz}, \varphi_{rz}) - 2\bar{\nu} \int_0^c (r\varphi_r)_r \varphi_{zz} \, \mathrm{d}r.$$

Let

$$I = (1-\nu_{TT}) \int_0^c (r\varphi_r)_r \varphi_{zz} \, \mathrm{d}r. \tag{6.15}$$

Then

$$-\frac{|\bar{\nu}|}{1-\nu_{TT}} [2(\varphi_{rz}, \varphi_{rz}) + 2|I|] \leqslant -\frac{2\bar{\mu}}{\pi} U'(z) + \frac{2\bar{\mu}}{\pi} V'(z) \leqslant \frac{|\bar{\nu}|}{1-\nu_{TT}} [2(\varphi_{rz}, \varphi_{rz}) + 2|I|]. \tag{6.16}$$

We now estimate $|I|$. Define the function ψ by

$$\varphi_{zz}(r, z) = \varphi_{zz}(c, z) + \psi(r, z). \tag{6.17}$$

Since φ_r vanishes at $r = c$, (6.15) can be written as

$$I = (1-\nu_{TT}) \int_0^c (r\varphi_r)_r \psi \, \mathrm{d}r. \tag{6.18}$$

Application of the Schwarz inequality and the second of (6.5) then yields the estimate

$$|I|^2 \leqslant (1-\nu_{TT})(\varphi_{zz}, \varphi_{zz}) \int_0^c \frac{1}{r} (r\varphi_r)_r^2 \, \mathrm{d}r. \tag{6.19}$$

Using (6.19) and the inequality $|xy| \leqslant \frac{1}{2}(x^2 + y^2)$ in (6.7), we readily obtain

$$2(\varphi_{rz}, \varphi_{rz}) + 2|I| \leqslant \frac{1}{\alpha} \left[-\frac{2\bar{\mu}}{\pi} V'(z) \right], \tag{6.20}$$

where α is defined in (6.10). Substitution of (6.20) in (6.16) and integrating from z to l then gives (6.9).

7. DECAY INEQUALITIES

In this section we shall show that the functional $V(z)$ introduced in (6.1) decays exponentially with distance z from the loaded end of the cylinder. The decay inequality has the form

$$V(z) \leqslant 2V(0)\exp(-2kz), \tag{7.1}$$

where the decay constant k depends on the elastic constants and is characterized as a root of a transcendental equation involving Bessel functions.

The steps in the proof are completely analogous to those used in [6]. Firstly we define the functional S by

$$S(z) = V(z) + 2k \int_z^l V(\zeta) \, \mathrm{d}\zeta, \quad 0 \leqslant z \leqslant l, \tag{7.2}$$

where k is an arbitrary positive constant. Then, using (7.2), (6.7), (6.8)

$$\begin{aligned} S'(z)+2kS(z) &= V'(z)+4k^2\int_z^l V(\zeta)\,\mathrm{d}\zeta \\ &= -\frac{\pi}{2\bar{\mu}}\left\{(\varphi_{zz},\varphi_{zz})+2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_{rz},\varphi_{rz})+A\int_0^c\frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r\right. \\ &\qquad\left.-4k^2(\varphi_z,\varphi_z)+4k^2(\varphi,\varphi_{zz})-4k^2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_r,\varphi_r)\right\} \\ &= -\frac{\pi}{2\bar{\mu}}\left\{(\varphi_{zz}+2k^2\varphi,\varphi_{zz}+2k^2\varphi)+A\int_0^c\frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r\right. \\ &\qquad -4k^2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_r,\varphi_r)-4k^4(\varphi,\varphi) \\ &\qquad\left.+2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_{rz},\varphi_{rz})-4k^2(\varphi_z,\varphi_z)\right\}. \end{aligned} \tag{7.3}$$

By virtue of (6.3) and the second of (6.5) we have $(\varphi_{zz}+2k^2\varphi, 1)=0$ and therefore

$$(\varphi_{zz}+2k^2\varphi,\varphi_{zz}+2k^2\varphi)\geqslant 0, \tag{7.4}$$

so that (7.3) implies

$$\begin{aligned} S'(z)+2kS(z)\leqslant -\frac{\pi}{2\bar{\mu}}&\left\{A\int_0^c\frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r-4k^2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_r,\varphi_r)\right. \\ &\left.-4k^4(\varphi,\varphi)+2\left(\frac{\bar{\mu}/\mu-\bar{\nu}}{1-\nu_{TT}}\right)(\varphi_{rz},\varphi_{rz})-4k^2(\varphi_z,\varphi_z)\right\}. \end{aligned} \tag{7.5}$$

Our aim now is to find a positive value of k such that the right hand side of (7.5) is non-positive. To this end, we recall here two lemmas concerning eigenvalue problems which have been discussed in [6].

LEMMA I. *Let $\psi\in C^1[0,c]$ and suppose that $\psi(c)=0$, $\psi(r)=O(r^2)$, $\psi_r(r)=O(r)$ as $r\to 0$. Then*

$$\int_0^c r^{-1}\psi_r^2\,\mathrm{d}r\geqslant\lambda_{\mathrm{I}}\int_0^c r^{-1}\psi^2\,\mathrm{d}r, \tag{7.6}$$

where λ_{I} is the smallest eigenvalue of the problem

$$(r^{-1}\tilde{\psi}_r)_r+\lambda r^{-1}\tilde{\psi}=0 \quad on \quad 0<r\leqslant c, \tag{7.7}$$

$$\tilde{\psi}=O(r) \quad as \quad r\to 0, \qquad \tilde{\psi}(c)=0. \tag{7.8}$$

LEMMA II. *Let $\theta\in C^1[0,c]$ and suppose that $(\theta,1)=0$. Then*

$$\int_0^c r\theta_r^2\,\mathrm{d}r\geqslant\lambda_{\mathrm{II}}(\theta,\theta), \tag{7.9}$$

where λ_{II} is the smallest positive eigenvalue of the problem

$$(r\tilde{\theta}_r)_r+(1-\nu_{TT})\lambda r\tilde{\theta}=0 \quad on \quad 0\leqslant r\leqslant c, \tag{7.10}$$

$$\tilde{\theta}_r(c)+\lambda c\tilde{\theta}(c)=0, \qquad \tilde{\theta}(r)=O(1) \quad as \quad r\to 0. \tag{7.11}$$

By using the lemmas, it is easily verified that

$$\int_0^c \frac{1}{r}(r\varphi_r)_r^2\, dr \geqslant \frac{\lambda_{\mathrm{I}}}{1-\nu_{TT}}(\varphi_r, \varphi_r), \tag{7.12}$$

$$(\varphi_{zr}, \varphi_{zr}) \geqslant (1-\nu_{TT})\lambda_{\mathrm{II}}(\varphi_z, \varphi_z), \tag{7.13}$$

and

$$(\varphi_r, \varphi_r) \geqslant (1-\nu_{TT})\lambda_{\mathrm{II}}(\varphi, \varphi). \tag{7.14}$$

From (7.12), (7.13), (7.5) and (6.6)

$$S'(z) + 2kS(z) \leqslant -\frac{\pi}{2\bar{\mu}}\left\{\left[\frac{\lambda_{\mathrm{I}} A}{1-\nu_{TT}} - 4k^2\left(\frac{\bar{\mu}/\mu - \bar{\nu}}{1-\nu_{TT}}\right)\right](\varphi_r, \varphi_r) - 4k^4(\varphi, \varphi) + [2(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}} - 4k^2](\varphi_z, \varphi_z)\right\}. \tag{7.15}$$

If k is now required to satisfy

$$\lambda_{\mathrm{I}} A - 4k^2(\bar{\mu}/\mu - \bar{\nu}) > 0, \tag{7.16}$$

(7.14) may be used in (7.15) to obtain

$$S'(z) + 2kS(z) \leqslant -\frac{\pi}{2\bar{\mu}}\{[(\lambda_{\mathrm{I}} A - 4k^2(\bar{\mu}/\mu - \bar{\nu}))\lambda_{\mathrm{II}} - 4k^4](\varphi, \varphi) + [2(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}} - 4k^2](\varphi_z, \varphi_z)\}. \tag{7.17}$$

We now further restrict k to satisfy

$$[\lambda_{\mathrm{I}} A - 4k^2(\bar{\mu}/\mu - \bar{\nu})]\lambda_{\mathrm{II}} - 4k^4 \geqslant 0, \tag{7.18}$$

$$2(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}} - 4k^2 \geqslant 0, \tag{7.19}$$

so that (7.17) becomes

$$S'(z) + 2kS(z) \leqslant 0,$$

from which

$$S(z) \leqslant S(0)\exp(-2kz). \tag{7.20}$$

The positive root of the polynomial in k in (7.18) is given by

$$k^* = \{\tfrac{1}{2}[\sqrt{\{(\bar{\mu}/\mu - \bar{\nu})^2\lambda_{\mathrm{II}}^2 + \lambda_{\mathrm{I}}\lambda_{\mathrm{II}} A\}} - (\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}}]\}^{1/2}. \tag{7.21}$$

Thus the largest value of k satisfying the three restrictions (7.16), (7.18), (7.19) is given by

$$k = \min\left\{\frac{1}{2}\left(\frac{\lambda_{\mathrm{I}} A}{\bar{\mu}/\mu - \bar{\nu}}\right)^{1/2}, \left[\frac{(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}}}{2}\right]^{1/2}, k^*\right\}. \tag{7.22}$$

From (7.21), it is easy to show that k^* is always less than or equal to the first member of (7.22), so that (7.22) may be written as

$$k = \min\left\{\left[\frac{(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}}}{2}\right]^{1/2}, k^*\right\}. \tag{7.23}$$

Alternatively, we may write (7.23) as

$$k = \begin{cases} k^* & \text{if} \quad \lambda_{\mathrm{I}} A < 3(\bar{\mu}/\mu - \bar{\nu})^2 \lambda_{\mathrm{II}} \\ \left[\dfrac{(\bar{\mu}/\mu - \bar{\nu})\lambda_{\mathrm{II}}}{2}\right]^{1/2} & \text{if} \quad \lambda_{\mathrm{I}} A \geqslant 3(\bar{\mu}/\mu - \bar{\nu})^2 \lambda_{\mathrm{II}}. \end{cases} \tag{7.24}$$

Thus we have established the exponential decay inequality (7.20) for $S(z)$, with k given by (7.24). Finally, by using exactly the same argument as that used in [2,6] it is easily shown that (7.20) implies the exponential decay inequality (7.1) for $V(z)$.

8. ENERGY AND STRESS ESTIMATES

On using the right hand inequality of (6.9), it follows from the exponential decay law (7.1) that the strain energy $U(z)$ satisifies

$$U(z) \leqslant 2\left(1 + \frac{|\bar{\nu}|}{(1 - \nu_{TT})\alpha}\right) V(0)\exp(-2kz), \qquad 0 \leqslant z \leqslant l, \tag{8.1}$$

where the positive constant α is defined in (6.10). If α is such that $(1 - \nu_{TT})\alpha - |\bar{\nu}| > 0$, then the left hand inequality of (6.9) may be used in (8.1) to yield the alternative estimate

$$U(z) \leqslant 2\left(\frac{(1 - \nu_{TT})\alpha + |\bar{\nu}|}{(1 - \nu_{TT})\alpha - |\bar{\nu}|}\right) U(0)\exp(-2kz), \qquad 0 \leqslant z \leqslant l. \tag{8.2}$$

Either (8.1) or (8.2) show that the strain energy decays exponentially away from the loaded end of the cylinder.

In several earlier works on Saint-Venant's principle in isotropic elasticity, pointwise estimates for the stress components at interior points in terms of the strain energy were obtained by using various mean value theorems of the theory of elasticity [21]. It may be possible to establish analogous mean value theorems for the present anisotropic case. An alternative approach is to use a mean value theorem based on the fourth order elliptic equation (3.3)† for the Lekhnitskii stress function Φ, which in turn leads to mean value theorems for each of the cylindrical stress components τ. In this way, we obtain (at an interior point (r, θ, z) of the cylinder) an inequality of the form

$$|\tau(r, z)| \leqslant K_1 \delta^{-3/2} \sqrt{[U(z - \delta)]}, \qquad 0 \leqslant r < c, \quad 0 < z < l, \tag{8.3}$$

where δ is the distance from the point (r, θ, z) to the boundary of the cylinder and K_1 is a constant depending on the elastic constants.

The estimate (8.3) is not valid for boundary points and clearly deteriorates as $\delta \to 0$. For isotropic elasticity Roseman[4] has developed an alternative approach which does not have this limitation.

The inequality (8.3), in conjunction with (8.1) or (8.2), shows that the stresses satisfy an inequality of the type

$$|\tau(r, z)| \leqslant K \exp(-kz), \qquad 0 < z < l, \tag{8.4}$$

where K is a constant‡ and k is given by (7.24). Thus the decay constant k, which clearly

† For a discussion of such mean value theorems, see paragraph 29 of Miranda[22].

‡ The quantities $U(0)$, $V(0)$ appearing in (8.1), (8.2) may be estimated by application of appropriate minimum principles, (cf.[2,3]) but we will not pursue this here.

depends on the elastic constants, provides a lower bound on the rate of exponential decay of stresses away from the loaded end of the cylinder. Our main interest here is in investigating this decay rate, particularly its dependence on the degree of anisotropy of the material and it is to this topic that we now turn.

9. THE DECAY CONSTANT. COMPARISON WITH OTHER WORK

The eigenvalue λ_{I} for the problem (7.7), (7.8) is given by

$$\lambda_{\mathrm{I}} = s^2/c^2 \tag{9.1}$$

where $s = 3.8317$ is the smallest zero of $J_1(s)$, where J_n denotes the Bessel function of order n. From the eigenvalue problem (7.10), (7.11), λ_{II} is given by

$$\lambda_{\mathrm{II}} = t^2/c^2 \tag{9.2}$$

where t is the smallest positive root of

$$-(1-\nu_{TT})^{1/2}J_1[(1-\nu_{TT})^{1/2}t] + tJ_0[(1-\nu_{TT})^{1/2}t] = 0. \tag{9.3}$$

We write (9.3) in the form

$$\mathscr{T}_1[(1-\nu_{TT})^{1/2}t] = 1-\nu_{TT}, \tag{9.4}$$

where $\mathscr{T}_1(s) \equiv sJ_0(s)/J_1(s)$ is the Onoe function of the first kind[23]. Clearly λ_{I} is a fixed quantity for all bodies, while λ_{II} depends only on the transverse Poisson's ratio ν_{TT}. Once λ_{I}, λ_{II} are known, the decay rate k is computed from (7.24).

As mentioned in the introduction here, the axisymmetric end problem for *isotropic* circular cylinders has been the subject of numerous investigations. However, the author is aware of only two such studies[16,17] for the case of a transversely isotropic cylinder. The work [16] is not of interest for our purposes here. In [17], as in many of the works on the isotropic problem, the semi-infinite cylinder is treated and an "exact solution" is found in the form of an infinite series of eigenfunctions. In this mode of approach, an exponentially decaying form of solution is *assumed* to begin with—the exact rate of decay is then found as a root of a transcendental equation for the eigenvalues (see equation (9.5) below). As pointed out in [17], the completeness issue for this method of solution has not been resolved.

Our approach here is rather to use widely applicable methods involving energy inequalities to prove that the stresses *do* decay exponentially away from the loaded end of a finite cylinder and to obtain a readily computable lower bound on this rate of decay.

The transcendental equation of [17] referred to above may be written†

$$\mathscr{T}_1(Kx) - K^2\mathscr{T}_1(x) + (1-\nu_{TT})(K^2-1) = 0, \tag{9.5}$$

where $\mathscr{T}_1$ is the Onoe function introduced above and $K = c_2/c_1$, where c_1^2, c_2^2 were defined at the beginning of section 3 here as roots of the quadratic equation (3.5). According to [17], the decay rate for a cylinder of unit radius is given by the real part of x_1/c_1, where x_1 is the first non-zero root of (9.5). Numerical results were presented by Warren *et al.* in [17] for the two transversely isotropic materials magnesium and zinc. In Table 1 here, we compare the decay rate found from the data of [17] with ck, where k is computed from (7.24). We use values for the elastic constants given in [17]. From the table we see that for magnesium, k is about 0.5 the corresponding value obtained from the data of [17]; an analogous result was true in a similar comparison for the isotropic cylinder[6]. For zinc, the lower bound k is conservative by a factor of about 2.5.

† See equation (16) of [17].

Table 1. Decay constant k for magnesium and zinc

Material	ck, with k computed from (7.24)†	(Decay constant) × c, from [17]‡
magnesium	1.45	2.87
zinc	1.09	2.82

† Roots of (9.4) were obtained with the aid of the tables in [23].
‡ Computed to two decimal places from data in Table 1 of [17].

10. LIMIT FOR HIGHLY ANISOTROPIC MATERIALS

One of the prime motivations for the present study was to determine the influence of anisotropy on the decay of end effects in circular cylinders. In view of the current widespread work on elasticity problems for composite materials, it is of interest to examine the validity of Saint-Venant's principle for highly anisotropic materials. Indeed previous work[7,8] on the *plane* problem has demonstrated the significant influence of anisotropy in this context.

To investigate the behaviour of the decay constant k given by (7.24) as the degree of anisotropy E_L/E_T increases, we proceed as follows. We introduce the notation $\varepsilon = E_T/E_L$ so that as the degree of anisotropy increases, the dimensionless quantity ε tends to zero. Recalling the definitions of A and $\bar{\nu}$ from (3.9), we have in the new notation

$$A = \frac{\varepsilon(1 - \nu_{LT}^2 \varepsilon)}{1 + \nu_{TT}}, \qquad \frac{\bar{\mu}}{\mu} - \bar{\nu} = \frac{G_T}{G_{LT}} - \varepsilon \nu_{LT}, \tag{10.1}$$

where ν_{LT}, ν_{TT} are the Poisson ratios introduced previously. As we remarked at the beginning of section 9, the eigenvalue λ_{I} is independent of the elastic constants while λ_{II} depends only on the transverse Poisson's ratio ν_{TT}. Thus, for small ε, (7.24a) applies and $k = k^*$ where k^* is given by (7.21).

Using the expressions given in (10.1) here, it is easy to show from (7.21) that as $\varepsilon \to 0$

$$2k^2 = \left(\frac{G_T}{G_{LT}} - \varepsilon \nu_{LT}\right)\lambda_{\mathrm{II}}[(1 + O(\varepsilon))^{1/2} - 1] \tag{10.2}$$

and so, on using (9.2) we obtain

$$k = O\left(\frac{\varepsilon^{1/2}}{c}\right) \qquad \text{as} \quad \varepsilon \to 0, \tag{10.3}$$

where c is the radius of the cylinder.† Thus the decay law (8.4), with k given by (10.3) predicts a *slow* rate of exponential decay of stresses away from the loaded end, with *large* characteristic decay length $c/\varepsilon^{1/2}$. Consequently the energy approach used here anticipates disagreement with Saint-Venant's principle in the limit considered above.

In conclusion we note that by considering the asymptotic form of (9.5) in the limit as $\varepsilon \to 0$, one can verify that the exact decay rate behaves precisely as predicted by the expression (10.3).

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† cf. equation (3.10) of [8].

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Абстракт — В целью исследования затухания краевых эффектов в поперечно изотропном, круглом цилиндре, под влиянием осесимметрических краевых нагрузок, не вызывающих кручения, применяются неравенства затухания энергии. Получается нижний предел (в обозначениях упругих постоянных) для скорости экспоненциального затухания напряжений и сравнивается он с результатами других авторов. Для сильно анизотропной среды, предсказывается медленная скорость затухания, по крайной мере упреждающая разногласие с принципом Сен-Венана для этого случая.